STAT0041: Stochastic Calculus

Lecture 8 - Brownian Motion and Heat Semigroup

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Key concepts:

- *Markov properties;*
- *Markov semigroup;*
- *Heat semigroup.*

8.1 Markov properties of Brownian motion

We begin with a brief review of the Markov process.

Definition 8.1 (Markov process) *Let $\{X_t : t \in \mathcal{T}\}$ be a $\mathscr{F}_t$-adapted stochastic process on $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathrm{P})$ with state space $(E, \mathscr{E})$, then followings are equivalent:*

(1) X is a Markov process;

(2) For all $A \in \mathscr{E}, s < t \in \mathcal{T}$,

$$\mathrm{P}(X_t \in A \mid \mathscr{F}_s) = \mathrm{P}(X_t \in A \mid X_s);$$

(3) (standard approximation procedure) For all bounded measurable function f on E, $s < t \in \mathcal{T}$

$$\mathbb{E}[f(X_t)|\mathscr{F}_s] = \mathbb{E}[f(X_t)|X_s] \quad a.s.$$

Definition 8.2 (Transition function) *Let $(E, \mathscr{E})$ be a state space, we say $p(s, x; t, A)$, $s, t \in \mathcal{T}$, $x \in E$, $A \in \mathscr{E}$ is a transition function on $(E, \mathscr{E})$, if*

(1) For fixed s, t, x, $p(s, x; t, \cdot)$ is probability measure on $(E, \mathscr{E})$;

(2) For fixed s, t, A, $p(s, \cdot; t, A)$ is $\mathscr{E}$-measurable function;

(3) (Kolmogorov-Chapman equation) For any $s < t < u \in \mathcal{T}$, $x \in E$, $A \in \mathscr{E}$,

$$p(s, x; u, A) = \int_E p(s, x; t, \mathrm{d}y)\, p(t, y; u, A). \tag{8.1}$$

Further, if exist $p(t, x, A)$ satisfies for all $s \in \mathcal{T}$,

$$p(s, x; s + t, A) = p(t, x, A),$$

we say transition function p is time homogeneous.

It can be proved when certain conditions hold, there exists a family of *transition function*

$$p(s,x;t,A) := \mathrm{P}(X_t \in A | X_s = x)$$

denote the probability of starting from x at time s and transferring to A at time t.

Theorem 8.3 *Brownian motion is a Markov process with time homogeneous transition function*

$$p(t,x,y) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|y-x|^2}{2t}}. \tag{8.2}$$

8.2 Markov semigroup

For Markov process (X_t) with state space $(E, \mathscr{E})$, define:

$$(P_t f)(x) = \mathbb{E}[f(X_t)|X_0 = x] = \int f(y)p(t,x,y)dy.$$

where

$$P_t : \mathcal{M}_b(E) \to \mathcal{M}_b(E),\ f \mapsto P_t f$$

where $\mathcal{M}_b(E)$ is the set of all bounded measurable function on E. Then P_t has semigroup property:

$$P_{t+s} f = P_t \circ P_s f.$$

We say (P_t) is a *Markov semigroup*. Recall that for discrete time homogeneous Markov Chains,

$$P_t = (P_1)^t,\ t \in \mathbb{N}.$$

Once we know initial state and one step transition matrix P_1, the Markov chain is clearly defined. We also want to define a similar quantity for continuous time Markov process. Thus we introduce following definition.

Definition 8.4 (Infinitesimal generator) *Let P_t be a operator semigroup,*

$$\mathcal{L} f := \lim_{t \downarrow 0} \frac{P_t f - f}{t}. \tag{8.3}$$

We say $\mathcal{L}$ is the ***infinitesimal generator*** *of P_t.*[1]

Proposition 8.5 (Kolmogorov forward/backward equation) *For all $t \geq 0$, it holds that*

$$\partial_t P_t f = \mathcal{L} P_t f = P_t \mathcal{L} f.$$

[1] In order to guarantee the existence of the limit in (8.3), we need to consider $f \in \mathcal{D}(\mathcal{A}) := \{f \in \mathcal{M}_b(E), \lim_{t\downarrow 0} \frac{P_t f - f}{t}$ is exist$\}$

8.3 Brownian motion and heat semigroup

Define

$$K(t,x,y) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x-y|^2}{2t}}.$$

We called $K(t,x,y)$ *heat kernel* because for a continuous and bounded function $f : \mathbb{R}^d \to \mathbb{R}$,

$$u(x,t) = \int K(t,x,y) f(y) \mathrm{d}y$$

solves the *heat equation*

$$\frac{1}{2}\Delta_x u(x,t) - \frac{\partial}{\partial t} u(x,t) = 0, \quad u(x,0) = f(x)$$

Following proposition shows connection between Brownian motion and heat equation.

Proposition 8.6 *Let (B_t) be a Brownian motion, then for all continuous and bounded function $f : \mathbb{R}^d \to \mathbb{R}$*

$$P_t f(x) := \mathbb{E}[f(B_t)|B_0 = x] = \int p(t,x,y) f(y) \mathrm{d}y \tag{8.4}$$

where

$$p(t,x,y) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x-y|^2}{2t}}.$$

Since $P_t f(x)$ satisfies

$$\frac{\partial}{\partial t} P_t f(x) = \frac{1}{2}\Delta_x P_t f(x),$$

operator semigroup P_t defined in (8.4) is called **heat semigroup**. And the infinitesimal generator of Brownian motion is Laplace $\frac{1}{2}\Delta$.